# On the Effect of Channel Fading on Greedy Scheduling

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**Abstract**

Greedy Maximal Scheduling (GMS) is an attractive low-complexity scheme for scheduling in wireless networks. Recent work has characterized its throughput for the case when there is no fading/channel variations. This paper aims to understand the effect of channel variations on the relative throughput performance of GMS vis-a-vis that of an optimal scheduler facing the same fading. The effect is not a-priori obvious because, on the one hand, fading could help by decoupling/precluding global states that lead to poor GMS performance, while on the other hand fading adds another degree of freedom in which an event unfavorable to GMS could occur.

We show that both these situations can occur when fading is adversarial. In particular, we first define the notion of a *Fading Local Pooling factor (F-LPF)*, and show that it exactly characterizes the throughput of GMS in this setting. We also derive general upper and lower bounds on F-LPF. Using these bounds, we provide two example networks - one where the relative performance of GMS is worse than if there were no fading, and one where it is better.

**Index Terms**

Local Pooling factor, Greedy Maximal Scheduling, Throughput Region, Channel Fading.

## I. INTRODUCTION

This paper analytically investigates the effect of fading on the throughput performance of a natural and popular scheduling algorithm: Greedy Maximal Scheduling (GMS) [12], [4], [3], [5]. As with any scheduling algorithm, GMS is a way to determine which wireless links can transmit at any given time, based on their mutual interference characteristics and their current level of fading. In particular, GMS involves first associating a weight with each

link – which depends on the load of the link and its channel condition. Then, GMS involves iteratively turning on the heaviest link that does not interfere with links already turned on. This is repeated every time slot.

GMS has empirically shown to have very good throughput and delay performance; recent theoretical advances [8], [5], [15], [1], [7], [10] characterize its throughput. All of these works assume that there is no fading; ie that the rate a link can support is invariant as long as all the links that interfere with it are not simultaneously on. Our work investigates what happens to this performance in the more realistic setting with intrinsic channel fading as well. In particular, we compare the relative throughput of GMS as compared to that of an optimal scheduler.

Our results demonstrate that the effect of fading is quite subtle; in particular, in some instances fading can degrade the relative performance of GMS, while in other cases it can improve it. The former reflects the fact that fading provides an extra degree of freedom and complexity in the system, which GMS may not be able to handle as well as in a system without this fading. The latter reflects the, perhaps more subtle, fact that the sub-optimality of GMS (even without fading) is tied to the existence of special global system configurations that result in poor performance. The presence of fading "breaks up" these global configurations – not allowing them to occur too often – allowing GMS to perform relatively better.

Specifically, our contributions are as follows: For a given wireless network with fading channels,

1) We define a new quantity, called Fading-Local Pooling Factor (F-LPF), analogous to LPF defined in [5] that characterizes the performance of Greedy Maximal Scheduling (GMS) in wireless networks with fading channels. Furthermore, we show that Fading-LPF is a lower bound on the fraction of throughput that can be stabilizable by the GMS when the arrivals and channels are independent and identically distributed over time.

2) With arbitrary arrival and channel state process, we show that Fading-LPF is an upper bound on the fraction of throughput that can be stabilizable by the greedy schedule. More specifically, we construct an adversarial arrival and channel process with long term averages that lie outside the scaled throughput region and show that GMS policy cannot stabilize the queues.

3) We further provide lower and upper bounds on Fading-LPF that are easy to evaluate. We provide two example networks with specific fading structure and use the derived bounds to demonstrate that fading can either enhance or degrade the relative performance of GMS as compared to the non-fading scenario.

4) With fading, we can represent the channel model as a collection of global channel-states, where each state is associated with an independent set and an occurance probability. A natural question that arises is the following: Is the acheivable rate-region with fading simply the (channel-probability weighted) average of the per-state *scaled* rate regions, with the scaling parameter simply being the conventional LPF for each state? We show that this is in general not true. However, we derive a region that *can* be stabilized by the GMS in wireless networks with fading channels. This region is characterized based on the interference degree of the subgraphs (generated from original network) and the fading distribution.

*A. Related Work:*

Transmission scheduling has been a key challenge in modern wireless systems. The MaxWeight algorithm, proposed in [14], has been the inspiration for many approaches to address this in various wireless systems (see [4] for several variants). However, this algorithm suffers from centralization as well as computational complexity.

Thus, there has been significant research in finding sub-optimal (i.e., achieving a subset of the throughput region) distributed scheduling algorithms with low complexity. The authors in [12] propose one such policy called Greedy Maximal Scheduling, whose time complexity is linear in the number of links, and has a distributed implementation [8]. There are other sub-optimal, randomized algorithms that have been proposed with similar performance as GMS [11], [6].

The authors in [3] have been the first to study the performance of GMS under a general interference model. They have identified conditions (so called 'Local Pooling') under which there is no loss in the network throughput region with GMS. The notion of Local Pooling has been extended to a multi-hop regime by [15].

This condition being identified as too restrictive, the authors in [5] have defined a new quantity called Local Pooling Factor (LPF) that exactly characterizes the fraction of throughput region achieved by GMS, and show that over tree networks with a $K-$hop model for interference, GMS achieves the entire throughput region. Additional characterizations, including a per-link LPF [9] and bounds to characterize the stability region [10], have been proposed in literature.

The authors in [1] exactly characterize, using graph theoretic methods, the set of network graphs (with only the primary interference constraints) where GMS is optimal (LPF = 1). Finally, the authors in [7] have studied the performance of GMS with the SINR interference model, and have shown that GMS exhibits zero LPF in the worst case.

All the above results assume that there are no channel variations (fading). In this paper, we study the effect of channel variation on the performance of GMS.

## II. SYSTEM MODEL AND BACK GROUND

We consider a wireless network consisting of $K$ links labeled as $\{1, 2, 3, ..., K\}$. Let $\mathcal{K}$ denote the set of links in the network. Each link $l$ consists of a transmitter and receiver. We assume time to be slotted. Each time slot is composed of two parts. The first (control) part is reserved for making the transmission decision and second part for transmitting the packet. At time slot $t$, we denote the channel capacity of link by $C_l[t]$. We assume that the capacity varies from slot to slot, and is constant during a time slot. We consider collision interference/protocol model and denote the set of links that interfere with link $l$ by $\mathcal{I}_l$. We say that the transmission on link $l$ at time $t$ is successful, if no link in the $\mathcal{I}l$ transmits during the same time $t$. The maximum number of packets that can be successfully transmitted in time slot $t$ on link $l$ is bounded by $C_l[t]$.

We assume single hop flows in the network. Let $A_l[t]$ denote the number of packets that arrive at transmitter of link $l$ at time slot $t$. We assume that arrival processes is bounded and average rate of arrivals for link $l$ is denoted

by $\lambda_l$.

For simplicity we first consider ON/OFF channels (i.e $C_l[t] = 0$ or 1) and later show that our results can be extended to channels with finite number of channel states. For the ON/OFF setting, global state (GS) refers to specifying the set of links that are in 'ON' state. Let $GS(t)$ denote the set of links that are in 'ON' state in time slot $t$. Let $\pi(J)$ denote the fraction of time the network is in global channel state $J$, where links in set $J$ are 'ON' and links in the set $\mathcal{K} \backslash J$ are in 'OFF' state. Let $\boldsymbol{\pi} := \{\pi(J), J \subset \mathcal{K}\}$ denote the *fading structure*.

**Assumptions.** :

A1 (Long-term Averages): *We assume that the long-term time averages of arrivals and channel states satisfy the following:*

$$\frac{1}{T} \sum_{t=0}^{T} A_l[t] \to \lambda_l \quad as \quad T \to \infty. \tag{1}$$

*and*

$$\frac{1}{T} \sum_{t=0}^{T} \mathbf{1}_{GS(t)=J} \to \pi(J) \quad as \quad T \to \infty. \tag{2}$$

A2 (Randomness): *We assume that arrivals are mutually independent i.i.d processes with $\lambda_l = E[A_l[t]]$. Similarly the channels are independent across time and form a stationary process with $\pi(J) = E[\mathbf{1}_{GS(t)=J}]$.*

While both assumptions A1 and A2 specify the same long-term averages, we note that assumptions in A1 allow for arrival and channel state processes to be *dependent across time and across links* in a deterministic, and possibly *adversarial manner*. The necessity for the above sets of assumptions will be clear as we state our main results in Section III.

*A. Preliminaries*

As discussed earlier, there is a rich history of analysis of GMS algorithms for the non-fading case [3], [5], [9], [10], [1], [7]. In this section we build on this notation in literature to allow for time-varying (fading) channels.

We define Interference graph $\mathcal{IG}$ for a set of links as follows: Each link is represented by a node and an edge is drawn between two nodes if transmissions on the corresponding links in the original graph interfere with each other. This model captures many existing wireless models and is quite general. We define the Independent set on this graph as set of nodes with no edges between them. Let $Q_l[t]$ denote the number of packets present at the transmitter at time $t$ waiting to get scheduled on link $l$. Let $S_l[t] \in \{0, 1\}$ denote the schedule decision for link $l$ at time $t$. At each time $t$, a schedule $\vec{S}[t]$ is determined based on the global queue state and channel state information at time $t$, that is $(\vec{Q}[t]), \vec{C}[t])$. We also assume that arrivals occur at the end of time slot, thus we have the following queue dynamics:

$$Q_l[t+1] = (Q_l[t] - C_l[t]S_l[t])^+ + A_l[t], \tag{3}$$

where $a^+ = \max(0, a)$.

Given the arrival traffic rate $\{\lambda_l\}_{l\in\mathcal{L}}$ and a scheduling policy, we say that the network is *stable* under scheduling policy if the mean of the sum of queue lengths is bounded. We say that an arrival rate vector $\{\lambda_l\}_{l\in\mathcal{L}}$ is *supportable* if there exists any scheduling policy that can make the network stable. We call the set of all arrival vectors that are supportable by *throughput region* and denote it as $\Lambda_f$, where $f$ denotes that the channels are fading.

We say that a scheduling policy is throughput optimal if it can stabilize the network for all arrival rates inside the throughput region.

*Definition 1:* ([5]) The interference degree $d_I(l)$ of link $l$ is the maximum number of links in the set $\{l \cup \mathcal{I}_l\}$ that can be active at the same time with out interfering with each other. The interference degree $d_I(G)$ of a graph $G = \{V, E\}$ is the maximum interference degree across all its links in $E$

Consider a wireless system with 4 links. Let $\mathcal{I}_1 = \{2\}$, $\mathcal{I}_2 = \{1, 3, 4\}$, $\mathcal{I}_3 = \{2, 4\}$ and $\mathcal{I}_4 = \{2, 3\}$. The interference graph is shown in the Figure 1 with the corresponding $d_I(l)$. The interference degree of this example graph is 2.
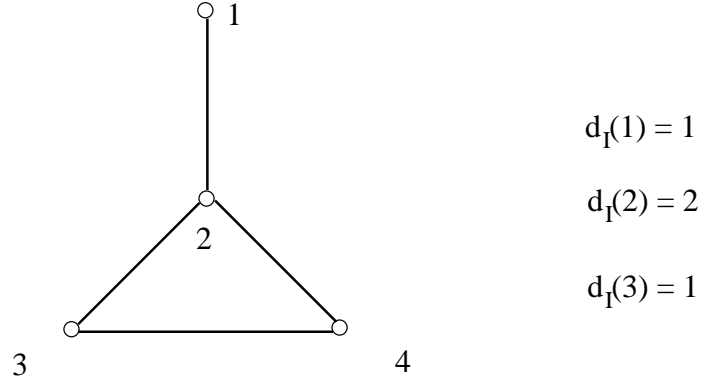


$$d_I(1) = 1$$
$$d_I(2) = 2$$
$$d_I(3) = 1$$

Fig. 1.   Interference Graph where nodes denote the links and edges denote the interference constraints.

*Definition 2:* Given an interference graph, an independent set corresponds to set of nodes (links in the original graph) such that there is no edge between any two nodes in the set (no two links interfere in the original graph). Further, it is maximal if it is not a subset of any other independent set. For a set of links $L$, define a matrix $M_L$ whose columns represent the maximal independent sets on the set $L$, with $|L|$ rows one for each link. We assume links are naturally ordered and rows in $M_L$ are assigned according to the defined order. For $J \subset L$, let $M_{J,L}$ denote the matrix with $|L|$ rows and is constructed from $M_J$ as follows: columns from $M_J$ are used and zero row vectors are added for links which do not belong to set $J$. Let $\mathcal{CH}(M_{J,L})$ denote the convex hull of all column vectors of matrix $M_{J,L}$.

For the above example with 4 links, let $J = \{1, 2, 3\}$ and $L = \{1, 2, 3, 4\}$, we have

$$M_J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$M_{J,L} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Note that the set $\Lambda_L := \{\vec{\lambda} : \vec{\lambda} < \vec{\mu}; \vec{\mu} \in \mathcal{CH}(M_L)\}$ characterizes the throughput region of set of $L$ links if no fading were present. We now define the throughput region with the *fading structure,*

*Definition 3:* The throughput region $\Lambda_f$ for a given network with fading pattern $\pi(J)$ is described as follows,

$$\Lambda_f = \left\{ \vec{\lambda} : \vec{\lambda} > 0, \vec{\lambda} \leq \sum_J \pi(J)\vec{\eta}_J \text{ where } \vec{\eta}_J \in \mathcal{CH}(M_{J,\mathcal{K}}) \right\}.$$

*Definition 4:* ([5]) The efficiency ratio $\gamma_{pol}^*$ under a given scheduling policy is defined as follows,

$$\gamma_{pol}^* = \sup\left\{ \gamma : \text{the policy can stabilize for all the arrival rate vectors } \lambda \in \gamma\Lambda_f \right\}.$$

*Definition 5:* Given $x(J) \in [0,1]$, we define a new region $\Lambda_f(\vec{x})$ as follows,

$$\Lambda_f(\vec{x}) = \left\{ \vec{\lambda} : \vec{\lambda} > 0, \vec{\lambda} \leq \sum_J x(J)\pi(J)\vec{\eta}_J \text{ where } \vec{\eta}_J \in \mathcal{CH}(M_{J,\mathcal{K}}) \right\}.$$

Note that throughput region is same as $\Lambda_f(1)$.

## B. GMS Algorithm [12]

We now describe the Greedy Maximal Scheduling(GMS) Algorithm. GMS essentially finds a maximal schedule in a greedy fashion. Each node in the interference graph is assigned weight equal to $f(Q_l(t)C_l(t))$, where $f(.)$ is a strictly increasing function that is zero at $0$ and tends to infinity as $Q_l(t)C_l(t) \to \infty$. It then proceeds as follows: it finds the node with maximum weight in the whole network and adds it to GMS schedule (ties are broken arbitrarily), it further discards all the neighboring nodes along with the selected node and repeats the above procedure on the reduced graph, till there are no more nodes left in the interference graph.

## III. MAIN RESULTS

In this paper, we characterize the performance of GMS algorithm for wireless networks with time-varying channels. We define the fading local pooling factor, $\sigma_L^*(\boldsymbol{\pi})$, for a set of links $L(\subseteq \mathcal{K})$ with fading structure $\boldsymbol{\pi}$ as follows:

$$\sigma_L^*(\boldsymbol{\pi}) = \inf\{\sigma : \exists \vec{\phi_1}, \vec{\phi_2} \in \Phi(L) \text{ such that } \sigma\vec{\phi_1} \geq \vec{\phi_2}\}, \tag{4}$$

where,

$$\Phi(L) = \{\vec{\phi} : \vec{\phi} = \sum_{J:J\subseteq\mathcal{K}} \pi(J)\vec{\eta}_J \text{ where } \vec{\eta}_J \in \mathcal{CH}(M_{J\cap L,L})\}, \tag{5}$$

and *Fading-Local Pooling Factor (F-LPF)* for a network $G$, $\sigma_G^*(\boldsymbol{\pi})$, with fading structure $\boldsymbol{\pi}$ as follows:

$$\sigma_G^*(\boldsymbol{\pi}) = \min_{L:L\subseteq\mathcal{K}} \sigma_L^*(\boldsymbol{\pi}), \tag{6}$$

Note that the above definition reduces to the known definition of LPF for a graph [5] when there is no fading, i.e, when $\pi(\mathcal{K}) = 1$.

The F-LPF can be understood as follows: Consider arrivals only to links of set $L$ (assume arrivals to other links are 0); when the links in set $J$ are 'ON' (others are 'OFF'), GMS will pick a maximal schedule among the 'ON' links, i.e. a column of $M_{J \cap L, L}$. Thus vector $\vec{\eta}_J$ is the long run average of these maximal schedules when system is in state $J$; so $\vec{\eta}_J \in \mathcal{CH}(M_{J \cap L, L})$. Thus $\Phi(L)$ is the set of all long-run average service vectors that could appear due to GMS when the arrivals are restricted only to set of links in $L$. For any two vectors $\vec{\phi}_1, \vec{\phi}_2 \in \Phi(L)$, it may thus happen that GMS results in $\vec{\phi}_2$ service vector, when it should have been $\vec{\phi}_1$ (for the optimal case). Thus $\sigma_L^*(\boldsymbol{\pi})$ is the worst possible ratio difference among all the possible service vectors of $\Phi(L)$.

**Dual Characterization and Implications:** In the same spirit as [3], [9], the *Fading-* Local Pooling Factor has a dual characterization, as noted in Lemma 3, and displayed below. The F-LPF, $\sigma_L^*(\boldsymbol{\pi})$, is given by the solution to the following optimization problem:

$$\sigma_L^*(\boldsymbol{\pi}) = \max_{x, a(J), b(J)} \sum_{J : J \subseteq L} \pi_L(J) a(J) \tag{7}$$

$$\text{s.t} : x' M_{J,L} \geq a(J) e' \quad \forall J \subseteq L$$

$$x' M_{J,L} \leq b(J) e' \quad \forall J \subseteq L$$

$$\sum_{J : J \subseteq L} \pi_L(J) b(J) = 1, \tag{8}$$

where $e$ is a column vector of all ones, $(\cdot)'$ is the vector transposition operation and $\boldsymbol{\pi}_L$ denotes the marginal distribution on set of links $L$ induced by $\boldsymbol{\pi}$.

Observe that each fading state $J$ *induces* a network defined by ON edges (i.e., all OFF links are removed from the network). Thus, one could ask if with fading channels, the F-LPF can be determined simply by computing the "standard" LPF (denoted by $\sigma^*(J)$) for each of these induced networks, and then averaging these quantities (weighted by the steady-state fractions of times for each of the fading states) over all possible fading states? In other words, *is the following true*?

$$\sigma_L^*(\boldsymbol{\pi}) \overset{?}{=} \sum_{J : J \subseteq L} \pi_L(J) \sigma^*(J)$$

where $\sigma^*(J)$ is the standard LPF [5] for the network that is induced by state $J$.

An important insight that emerges from the dual characterization is that such **averaging does necessarily not hold**, in particular because the possibly adversarial nature of the fading channel does not permit averaging. Note that the adversary *cannot* change the long-term fractions of the global states – it can merely change the temporal correlations. Inspite of this, averaging does not hold, as clearly shown in Example B in Section III-B).

In a tree network with fading as in Example B (see Section III-B), while the LPF for each state is '1', the F-LPF is less than $4/5$ which is lower than *any* convex averaging of the states! This discussion implies that the regular

LPF does not immediately extend to the case with fading. This motivates us to explicitly develop the local pooling factor in the presence fading, and understand its implications.

**Contributions:**

*A. Characterization in terms of F-LPF:*

Our first contribution, Theorem 1, characterizes the efficiency ratio of GMS algorithm in the presence of fading.

**Theorem 1.** *a) (Upper Bound) Under a given network topology and channel state distribution with Assumption A1 on the arrivals and fading channels, the efficiency ratio of GMS ($\gamma^*$) is less than or equal to $\sigma_G^*(\boldsymbol{\pi})$.*

*b) (Achievability) Under a given network topology and channel state distribution $\boldsymbol{\pi}$ with Assumption A2 on the arrivals and fading channels, the efficiency ratio of GMS ($\gamma^*$) is greater than or equal to $\sigma_G^*(\boldsymbol{\pi})$.*

*Implications:* The above result enables us to understand the performance of GMS compared to the optimal scheduler in the presence of fading. In particular, computing bounds on $\sigma_G^*(\boldsymbol{\pi})$ leads to insights on the positive and negative aspects of fading (discussed further in Theorems 2 and 3). Observe first that as long as the long-term averages on the arrivals and channels are satisfied (Assumption A1), we can construct an arrival and channel process that ensures that the efficiency *cannot* exceed the F-LPF $\sigma_G^*(\boldsymbol{\pi})$. Further, for *typical* arrival and channel processes with sufficient randomness (in this paper i.i.d. assumptions have been imposed, however this can be weakened), the converse holds wherein $\sigma_G^*(\boldsymbol{\pi})$ is achievable.

*Proof Discussion:* For the first part, we extend the ideas in [5], to construct an adversarial arrival and *fading* process pattern when arrival rates are outside the $(\sigma_G^*(\boldsymbol{\pi}) + \epsilon)\Lambda_f$ and show that a set of queues are unstable under GMS policy. For the second part, we use the approach in [3], [5] as follows: we show that if $\vec{\lambda}$ is inside $(\sigma_G^*(\boldsymbol{\pi}) - \epsilon)\Lambda_f$ then GMS policy can stabilize all the queues in the network. We look at the deterministic fluid limit of the system and exhibit a Lyapunov function whose drift is negative under the GMS policy. We have that fluid model is stable and therefore that the original system is stable.

**Theorem 2** (Upper Bound). *For every $J \subseteq \mathcal{K}$ and any $(\vec{\mu}_J, \vec{\nu}_J, H_J)$ such that $\vec{\mu}_J, \vec{\nu}_J \in \mathcal{CH}(M_J)$, $\vec{\nu}_J \leq H_J\vec{\mu}_J$, we have that*

$$\sigma_G^*(\boldsymbol{\pi}) \leq max_l \frac{\sum_{J \subseteq \mathcal{K}} \pi(J) H_J \mu_J(l)}{\sum_{J \subseteq \mathcal{K}} \pi(J) \mu_J(l)},$$

*where $\mu_J(l) = 0$ if $l \notin J$.*

*Implications:* While $\sigma_G^*(\boldsymbol{\pi})$ is defined only though an optimization problem, the upper bound permits an explicit solution. This bound is useful, as evidenced in Example B provided in Section III-B. In particular this upper bound is useful to illustrate that the F-LPF is not a simple convex combination of the standard LPF averaged over the fading states, and that adversarial fading can indeed worsen the performance of GMS.

*Proof Discussion:* Though the proof follows from straightforward algebraic computations, the value of the theorem lies in the smart selection of $(\vec{\mu}_J, \vec{\nu}_J, H_J)$ vectors that satisfy the inequality stated in the above theorem. In the

worst case the bound yields 1; however we can use the existing results in literature [1] to get good bounds. Thus, the tightness of the upper bound depend up on the ability to identify good vectors that satisfy the above constraints.

**Theorem 3** (Lower Bound).

$$\sigma_L^*(\boldsymbol{\pi}) \geq \frac{\sum_{J \subseteq L} \pi_L(J) n(M_J)}{\sum_{J \subseteq L} \pi_L(J) N(M_J)}, \tag{9}$$

where $n(M) = \min_j \sum_i M_{ij}, N(M) = \max_j \sum_i M_{ij}$. $\boldsymbol{\pi}_L$ denotes the marginal distribution on set of links $L$ induced by $\boldsymbol{\pi}$ and can be computed as follows,

$$\boldsymbol{\pi}_L(J) = \sum_{I:I \subseteq \mathcal{K}, I \cap L = J} \boldsymbol{\pi}(I)$$

.

*Implications:* The ability to compute a lower bound leads to the interesting observation that fading can help *improve* efficiency. This is because, by turning links 'OFF', fading "breaks up" some of the bad global states that can lead to poor GMS performance. This is explicitly brought out in Example A in the context of a six-link network.

*Proof Discussion:* The lower bound is derived using the dual formulation of the F-LPF, see (7). We find a point in the dual search space that satisfies all the constraints in the dual characterization, thus yielding a lower bound on the primal problem. Observe that $n(M_J)$ corresponds to the minimum number of links that needs to be 'ON' in any maximal schedule on set of $J$ links and $N(M_J)$ denotes the maximum number of links that could be 'ON' among all the maximal schedules on set of $J$ links. Thus, the lower bound can be computed easily and can be shown to be tight for some wireless networks. As an interesting aside, note that the lower bound provided is always better than the inverse of the interference degree of graph $G$ (see Corollary 1).

We now present two examples: A and B, one in which fading reduces the relative performance of GMS and the other in which fading enhances the relative performance of GMS respectively to illustrate the value of the above results.

*B. Examples: Benefit and Detriment with Fading*

**Example A: A network where fading structure improves the relative performance of GMS:** Consider a graph with six links $\mathcal{K} = \{a, b, c, d, e, f\}$. The interference graph for the six links is shown in the Figure III-B. Each link is either is state 'ON' or 'OFF'. We consider the following fading structure, $\boldsymbol{\pi}$, for $J \subseteq \mathcal{K}$

$$\pi(J) = p^{|J|}(1 - p)^{6-|J|},$$

where $|J|$ denotes the size of set $J$. Note that $p = 1$ corresponds to the no-fading case.

Using our results, we compute the lower bound and upper bounds on local pooling factor $\sigma_G^*(\boldsymbol{\pi})$ and is plotted in Figure III-B.

It is known [5] that the non-fading LPF for the above example is equal to 2/3. From the graph, we observe that for smaller values of $p$, F-LPF for above hexagon network with fading is greater than LPF with out fading
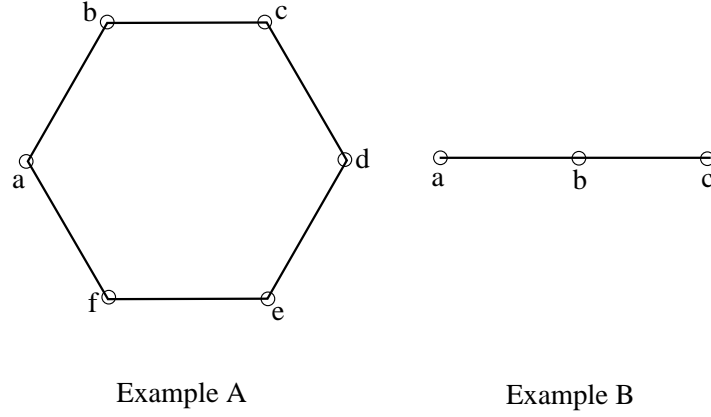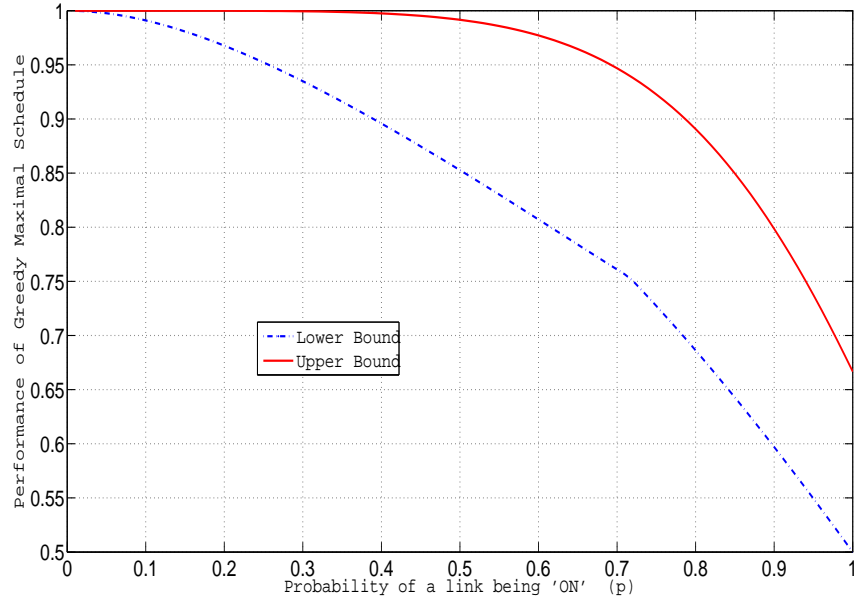
Fig. 2.  Interference graphs for the two example networks



Fig. 3.  Bounds on the fading local pooling factor for the Hexagon network

structure. As p tends to zero, the fraction of time network remains a cycle also tends to be small and it is known that GMS is optimal for tree networks. Therefore, it fits well with intuition to see that fading enhances the F-LPF for graphs with cycles.

**Example B: A network where fading structure worsens the relative performance of GMS:** Consider the graph with 3 links $a, b, c$ as shown above. The interference sets for each link is: $\mathcal{I}_a = \{b\}, \mathcal{I}_b = \{a, c\}$ and $\mathcal{I}_c = \{b\}$. We assume each link is either in state 'ON'(1) or 'OFF'(0). So the global channel state $'110'$ denotes that link $a$ and $b$ are in 'ON' state and link $c$ is in 'OFF' state. The fading structure is defined as follows: $\pi('110') = \pi('011') = \pi('111') = 1/3$.

For each global channel state, the possible maximal independent sets are as follows:

$$M_{ab,abc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$M_{bc,abc} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$M_{abc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Any vector that belongs to $\Phi(\{abc\})$ can be represented as follows,

$$\vec{\phi} = \frac{1}{3} M_{ab} [\alpha \, 1 - \alpha]' + \frac{1}{3} M_{bc} [\beta \, 1 - \beta]' + \frac{1}{3} M_{abc} [\gamma \, 1 - \gamma]'. \tag{10}$$

Let $\vec{\phi_1}$ be obtained using $(\alpha, \beta, \gamma) = (1, 0, 0)$ and $\vec{\phi_2}$ be obtained using $(\alpha, \beta, \gamma) = (1/2, 1/2, 3/4)$. Evaluating the above expression using the above values, we have $\vec{\phi_1} = \frac{1}{3}[1\,1\,1]'$ and $\vec{\phi_2} = \frac{5}{12}[1\,1\,1]'$. Observing the fact that $\frac{4}{5}\vec{\phi_2} = \vec{\phi_1}$, using Theorem 2, we have that local pooling factor for the wireless network with the above fading structure is less than or equal to $\frac{4}{5}$. But, it is known that the local pooling factor of GMS for tree networks (with no fading) is 1.

This result though sounds counter-intuitive, stems from the fact that we allow the fading to be arbitrary. Thus fading can act as adversary and as demonstrated, can degrade the performance of GMS algorithm.

*C. Characterization in terms of Interference degree*

So far, we have characterized the performance of GMS through a single scaling factor of the entire throughput region. Note that each fading state $J$ induces a network defined on the set of edges that are in 'ON' state and GMS can stabilize the network if arrivals are inside the region $\sigma^*(J)\Lambda_J$. It is natural to ask for the fading scenario, i.e. network with distribution $\pi(J)$, *if GMS could stabilize the region* $\sum_J \pi(J)\sigma^*(J)\Lambda_J$? We answer the above question in two parts.

In the first part, we show the interesting result that *GMS cannot stabilize* the above averaged region. In other words, there exists an arrival process with rate outside the region $\Lambda_f(\vec{x})$ for $x(J) = \sigma^*(J)$ (standard LPF) that can make the network unstable under GMS algorithm. We illustrate this using a simple example described below.

**Counter Example:** Consider the network with 3 nodes as in Example B. Note that the standard LPF [1] for all the three fading states is 1. Thus the region $\Lambda_f(\sigma^*(J))$ is exactly same as the actual throughput region $\Lambda_f$. However,

we have shown earlier that F-LPF is strictly less than $0.8$. Thus there exists an arrival process with rates outside the region $0.8\Lambda_f$ that cannot be stabilized by the greedy maximal schedule.

Given the previous negative result, in the second part we show that GMS can stabilize the region $\Lambda_f(\frac{1}{d_I(J)})$. Note that this region is strictly inside the region $\Lambda_f(\vec{x})$ with $x(J) = \sigma^*(J)$. More formally, our result is as follows:

**Theorem 4.** *Under a given network topology and channel state distribution with Assumption A1 on the arrivals and fading channels, GMS can stabilize the network if the arrival rates are inside the region $\Lambda_f(\vec{x})$, where $x(S) = \frac{1}{d_I(S)}$.*

*Implications:* The above theorem provides an elegant characterization of the rate region that can be stabilizable by the GMS algorithm. Also, we find that that the above region is *not a subset* of the achievable region stated in Theorem 1b (i.e $\sigma_G^*(\boldsymbol{\pi})\Lambda_f$). We illustrate the above observation through a simple example described below.

Consider the wireless network with 3 nodes and fading distribution similar to example B. Note that the interference degree for fading state $'110'$ is $d_I('110') = 1$, for state $'011'$ is $d_I('011') = 1$ and for the fading state $'111'$ is $d_I('111') = 0.5$. Any arrival rate vector that belongs to the new region defined using the interference degree can be expressed as below,

$$\vec{\lambda} = \frac{1}{3}M_{ab}[\alpha \, 1-\alpha]' + \frac{1}{3}M_{bc}[\beta \, 1-\beta]' + \frac{1}{3}\frac{1}{2}M_{abc}[\gamma \, 1-\gamma]', \tag{11}$$

where $\alpha, \beta$ and $\gamma$ are positive constants that are bounded by $1$. Using $(\alpha, \beta, \gamma) = (0,1,0)$, we have that rate vector $(0, \frac{5}{6}, 0)$ is inside the new region characterized by the interference degree. However, note that we have shown the F-LPF is upper bounded by $\frac{4}{5}$ for example B network. Thus, all arrival rates that are inside the region $\frac{4}{5}\Lambda_f$ satisfy the constraint that $\lambda_2 < \frac{4}{5}$ and hence rate vector $(0, \frac{5}{6}, 0)$ belongs to the new region and not the region characterized by F-LPF.

*Proof Discussion:* We consider the continuous time model with deterministic arrival and channel state processes. We then exhibit a Lyapunov function, sum of squares of queue lengths, whose derivative is strictly less than zero under the GMS policy whenever the arrival rate is strictly inside the new region. Therefore, the fluid model is stable and thus using the results from [2] we conclude that the original network model is stable.

## IV. PROOFS OF RESULTS

**Theorem (1).** *a) (Upper Bound) Under a given network topology and channel state distribution with Assumption A1 on the arrivals and fading channels, the efficiency ratio of GMS ($\gamma^*$) is less than or equal to $\sigma_G^*(\boldsymbol{\pi})$.*

*b) (Achievability) Under a given network topology and channel state distribution $\boldsymbol{\pi}$ with Assumptions A1 and A2 on the arrivals and fading channels, the efficiency ratio of GMS ($\gamma^*$) is greater than or equal to $\sigma_G^*(\boldsymbol{\pi})$.*

*Proof: The proof follows the method developed by the authors in [5], [3] for the non-fading case; however we have extended it to take in to account the fading structure. First, for the converse (to show instability for arrivals outside the stability region), we explicitly construct an adversarial channel variations pattern that satisfies the time-averages imposed by the fading assumption, and this is used in conjunction with the adversarial arrival*

*process. The achievability part is more straightforward – we augment the analysis in [3], [5] to include the fluid limit of the channel fading process.* We now provide the proof more detail:

**Proof (Theorem 1. a)**: The result follows from the following general lemma.

**Lemma 1.** *If there exists a subset of links $L(\subseteq \mathcal{K})$, a positive number $\sigma$ and two vectors $\vec{\mu}, \vec{\nu} \in \Phi(L)$ such that $\sigma \vec{\mu} > \vec{\nu}$, then for arbitrary small $\epsilon > 0$, there exists a traffic pattern with offered load $\vec{\nu} + \epsilon \vec{e}_L$ and a fading pattern, such that system is unstable under greedy maximal schedule.*

**Proof (Lemma 1)**: The idea of the proof is as follows – we construct a traffic pattern and channel variations pattern with offered load $\vec{\nu} + \epsilon \vec{e}_L$ and show that under this traffic/channel fading pattern, the queue lengths go to infinity under GMS, thus making the system unstable.

*As remarked earlier, this proof technique was introduced in [5], where authors only needed to construct adversarial arrival process that makes the queues in the system to overflow. However, in our setting, we need to account for the fading process and construct both arrival and channel fading pattern that makes the network unstable.*

Since $\vec{\nu} \in \Phi(L)$, there exist vectors $\vec{w}^J$ such that $\vec{\nu}$ can be expressed as,

$$\vec{\nu} = \sum_{J \subseteq L} \pi_L(J) \Big( M_{J,L} \vec{w}^J \Big). \tag{12}$$

Fix $\delta > 0$, we then find a vector $\vec{r}^J$ in the set of rational numbers, $\mathbb{Q}$, such that $\|\vec{r}^J - \vec{w}^J\| < \delta$.

Assume packets arrive to a link at beginning of the time slot. Let the queues of all the links in $L$ are empty at $t = 0$. Let $T_J$ be the smallest integer such that for all $i$, $r_i^J T_J$ is an integer. Let $t_i^J = r_i^J T_J$. Also, there exists integers $n_1, n_2, ... n_{2^L}$ such that

$$\Big| \frac{n_J T_J}{\sum_{S:S \subseteq L} n_S T_S} - \pi_L(J) \Big| \leq \frac{\delta}{2^L}. \tag{13}$$

Let us define $\tilde{\pi}_L(J) \in \mathbb{Q}$ as follows,

$$\tilde{\pi}_L(J) := \frac{n_J T_J}{\sum_{S \subseteq L} n_S T_S}. \tag{14}$$

Using the rational quantities $\tilde{\pi}_L(J)$ and $\vec{r}^J$, we define $\vec{\nu}^r$ as follows,

$$\vec{\nu}^r = \sum_{J:J \subseteq L} \tilde{\pi}_L(J) \big( M_{J,L} \vec{r}^J \big). \tag{15}$$

Consider a total time period of $\sum_J n_J T_J$. We assume that channel state remains in $J$ state for $T_J$ time slots (denoted as a time frame). It is easy to observe that with the above described fading pattern, we achieve the same channel state distribution as $\tilde{\pi}_L(J)$ on links of set $L$. We now describe the arrival pattern for $T_J$ time slots when the channel is in state $J$.

Assume that all the queue lengths (of links in $L$) are equal at the beginning of $T_J$ time slots. We now construct arrival pattern that keeps the queue lengths of all links in set $L$ equal at the end of $T_J$ time slots under the GMS policy. The arrival process is as follows:

1) The time frame of $T_J$ slots is further divided in to $t_1^J, t_2^J, ....t_{|IS^J|}$ time slots, where $t_i^J = r_i^J T_J$ and $|IS^J|$ denotes the number of columns in $M_J$.

2) During the $t_i^J, i \neq |IS^J|$ time slots, apply one packet to each link that is 'ON' in the $i^{th}$ column of $M_J$. For the last $t_{|IS^J|}^J$ time slots, apply one packet to each link that is ON in the last column of $M_J$ at the beginning of the time slot except for the last one time slot. For the last one time slot, with probability $1 - \epsilon$ we do the same as described before and with probability $\epsilon$, we apply two packets to each link that is ON in the last column of $M_J$ and 1 packet to rest of links in $L$.

*Note that the arrival process is modified compared to one proposed in [5] so as to ensure that all queues remain equal after $T_J$ time slots.*

It is now easy to see that at the end of $T_J$ time slots, all the queue lengths are equal and increase by 1 with probability $\epsilon$. Thus the above arrival and channel variation pattern make the system unstable under GMS schedule. We now show that the arrival rate is same as $\vec{\nu} + \epsilon \vec{e}_L$.

Let $\vec{e}_i$ denote the vector of all zeros except for $i$ th position which is set to one. Let $\sum_J = \sum_{J \subseteq L}$ for the remaining part of the proof. For the constructed adversarial arrival process, the arrival rate is given by the following,

$$\vec{\lambda}_{\text{adv}} = \frac{\sum_J n_J(\sum_{i=1}^{|IS^J|} t_i^J M_J \vec{e}_i + \epsilon \vec{e})}{\sum_J n_J(\sum_{i=1}^{|IS^J|} t_i^J)} \tag{16}$$

Rewriting the above expression in terms of $\tilde{\pi}_L(J)$, we have that

$$\vec{\lambda}_{\text{adv}} = \sum_J \tilde{\pi}_L(J)\left(\sum_{i=1}^{|IS^J|} r_i^J M_J \vec{e}_i\right) + \epsilon\left(\sum_J \frac{\tilde{\pi}_L(J)}{T_J}\right)\vec{e} \tag{17}$$

Thus we have,

$$\vec{\lambda}_{\text{adv}} = \sum_J \tilde{\pi}_L(J)\left(M_{J,L}\vec{r}^J\right) + \epsilon\left(\sum_J \frac{\tilde{\pi}_L(J)}{T_J}\right)\vec{e} \tag{18}$$

We choose small enough $\delta$ so that the arrival rate is strictly less than $\vec{\nu} + \epsilon \vec{e}_L$.

**Proof (Theorem 1. b)**: *This proof is a simple extension of that in [5], [3], however modified to include the fluid limit arising due to the channel fading process. Thus, we have provided a detailed sketch and refer to [5], [3] for full details.*

We consider the fluid limit of the queuing process and we provide a Lyapunov function and show negative drift under GMS schedule whenever arrival rate $\vec{\lambda} \in (\sigma_G^*(\pi) - \epsilon)\Lambda_f$.

Consider a sequence of systems $\frac{1}{n}\vec{Q}^n(nt)$ (scaled in time and space by a factor of $n$), where $\vec{Q}^n(.)$ denotes the queue lengths of original system, satisfying $\sum Q_l^n(0) \leq n$ at time $t = 0$. Let us index the sequence of systems by $n = \{1, 2, ....\}$. We apply the same arrival processes to all the above defined systems (i.e $\vec{A}^n(.) = \vec{A}(.)$) and assume that queues are served according to greedy maximal schedule. Let $\vec{A}^n(t)$ and $\vec{D}^n(t)$ denote the cumulative arrival and departure process of system $n$ up to time $t$.

Using the results from [2], it can be shown that the sequence of processes $(\vec{Q}^n(.), \vec{A}^n(.), \vec{D}^n(.))$ as $n \to \infty$ converges to a fluid limit almost surely along a subsequence $\{n_k\}$ in the topology of uniform convergence over compact sets,

$$\frac{1}{n_k}A_l^{n_k}(n_k t) \rightarrow \lambda_l t, \tag{19}$$

$$\frac{1}{n_k}D_l^{n_k}(n_k t) \rightarrow \sum_J \pi(J)\Big(\int_0^t \mu_l^J(s)ds\Big), \tag{20}$$

$$\frac{1}{n_k}Q_l^{n_k}(n_k t) \rightarrow q_l(t). \tag{21}$$

Also, the fluid limits $(q_l(t), \mu_l^J(t))$ satisfy the following equality:

$$q_l(t) = q_l(0) + \lambda_l t - \sum_J \pi(J)\Big(\int_0^t \mu_l^J(s)ds\Big). \tag{22}$$

Moreover, fluid limits are absolutely continuous, and at regular times $t$ (i.e., those points in time where the derivatives exist) we have the following condition satisfied:

$$\frac{d}{dt}q_l(t) = \begin{cases} \lambda_l - \mu_l(t) & \text{if } q_l(t) > 0 \\ (\lambda_l - \mu_l(t))^+ & \text{if } q_l(t) = 0, \end{cases}$$

where $\mu_l(t) = \sum_J \pi(J)\mu_l^J(t)$ satisfies the GMS properties. Let $L_0$ denote the set of links with the longest queues at time $t$,

$$L_0(t) = \big\{i \in K | q_i(t) = \max_{j \in K} q_j(t)\big\} \tag{23}$$

Let $L(t)$ denote the set of links with the largest derivative of queue length among the links in $L_0(t)$,

$$L(t) = \big\{i \in L_0(t) | \frac{d}{dt}q_i(t) = \max_{i \in L_0(t)} \frac{d}{dt}q_i(t)\big\} \tag{24}$$

**Lemma 2.** *Under the greedy maximal schedule, the service rate satisfies $\vec{\mu}(t)|_{L(t)} \in \Phi(L(t))$, where $\vec{u}|_L$ denotes the projection of vector on $u$ on to set of links $L$.*

The proof of the above lemma is similar to one in [3], [5] and is presented in appendix. The idea, roughly is that, queues in the set $L(t)$ will remain the longest for small enough amount of time past $t$ and GMS picks the maximal schedule restricted to links in $L(t)$ that are in 'ON' state.

Since the arrival rates are strictly with in $\sigma_L^*(\pi)\Lambda_f$, there exists a service vector $\vec{\nu} \in \Phi(L)$ and $\vec{\nu} < \sigma_L^*(\pi)\Lambda_f$ such that $\vec{\lambda}(L) \leq \vec{\nu}$, where $\vec{\lambda}(L)$ is projection of arrival vector on to the set $L$. Given any two vectors in set $\Phi(L)$, note that one vector never dominates the other one in all the dimensions by a factor more than $\sigma_L^*(\pi)$. Therefore we have that $\frac{d}{dt}\max_{i \in L(t)}q_i(t)$ is strictly negative when ever $\max q_i(t) > 0$.

Let $V(t) = \max q_l(t)$ denote the Lyapunov function used for the fluid system. Since we have a negative drift for the Lyapunov function, using the results from [2], we have that fluid system is stable (i.e there exists $t_0 > 0$ such that $q_l(t) = 0 \forall t > t_0$). Therefore from [2], we have that the queues in the original queuing system are stable. ∎

**Theorem (2).** *For every $J \subseteq \mathcal{K}$ and any $(\vec{\mu}_J, \vec{\nu}_J, H_J)$ such that $\vec{\mu}_J, \vec{\nu}_J \in \mathcal{CH}(M_J)$, $\vec{\nu}_J \leq H_J \vec{\mu}_J$, we have that*

$$\sigma_G^*(\boldsymbol{\pi}) \leq max_l \frac{\sum_{J \subseteq \mathcal{K}} \pi(J) H_J \mu_J(l)}{\sum_{J \subseteq \mathcal{K}} \pi(J) \mu_J(l)},$$

*where $\mu_J(l) = 0$ if $l \notin J$.*

*Proof:*

Since $(\vec{\mu}_J, \vec{\nu}_J, H_J)$ satisfy the inequality,

$$\vec{\nu}_J \leq H_J \vec{\mu}_J \tag{25}$$

Summing over all subsets with positive scaling constants $\pi(J)$,

$$\sum_J \pi(J) \nu_J(l) \leq \sum_J \pi(J) \big( H_J \mu_J(l) \big) \tag{26}$$

Using the maximum constant over all the inequalities, we have the following,

$$\sum_J \pi(J) \vec{\nu}_J \leq \Big( max_l \frac{\sum_J \pi(J) H_J \mu_J(l)}{\sum_J \pi(J) \mu_J(l)} \Big) \sum_J \pi(J) \vec{\mu}_J \tag{27}$$

By observing the fact that $(\sum_J \pi(J) \vec{\nu}_J, \sum_J \pi(J) \vec{\mu}_J)$ belong to the $\Phi(\mathcal{K})$, we have the result. ∎

**Theorem (3).**

$$\sigma_L^*(\boldsymbol{\pi}) \geq \frac{\sum_{J \subseteq L} \pi_L(J) n(M_J)}{\sum_{J \subseteq L} \pi_L(J) N(M_J)}, \tag{28}$$

*where $n(M) = min_j \sum_i M_{ij}, N(M) = max_j \sum_i M_{ij}$ and $\boldsymbol{\pi}_L$ denotes the marginal distribution on set of links $L$ induced by $\boldsymbol{\pi}$.*

*Proof:* We first state a lemma that describes the dual problem that finds the fading Local Pooling Factor as the optimal solution. The dual characterization of Local Pooling Factor was presented previously in [3], [9]. We now provide such characterization for F-LPF in Lemma 3 by generalizing the arguments in [9]. In particular, the multiple global channel states due to fading each induce a different constraint – combining all of these appropriately while satisfying the long-term average fractions $\{\pi_L(J)\}$ results in a $\max \min$ problem, as detailed below. This result is used to derive the lower bound.

**Lemma 3.** *The following optimization problem characterizes $\sigma_L^*(\boldsymbol{\pi})$ :*

$$\sigma_L^*(\boldsymbol{\pi}) = \max \sum_{J:J \subseteq L} \pi_L(J) a(J)$$

$$s.t : x' M_{J,L} \geq a(J) e' \quad \forall J \subseteq L$$

$$x' M_{J,L} \leq b(J) e' \quad \forall J \subseteq L$$

$$\sum_{J \subseteq L} \pi_L(J) b(J) = 1$$

*Proof:* Consider the definition of $\sigma_L^*(\pi)$ in (4). The corresponding optimization problem is given by:

$$\inf \quad \sigma$$

$$\text{s.t} : \sigma \sum_{J \subseteq L} \pi_L(J) M_{J,L} \vec{\alpha}(J) \geq \sum_{J \subseteq L} \pi_L(J) M_{J,L} \vec{\beta}(J)$$

$$\|\vec{\alpha}(J)\| = 1 \qquad \forall \quad J \subseteq L$$

$$\|\vec{\beta}(J)\| = 1 \qquad \forall \quad J \subseteq L$$

$$\vec{\alpha}(J), \vec{\beta}(J) \geq 0$$

where $\|.\|$ is defined as the sum of all the elements of the vector. Let us define a new variable $\vec{\gamma}(J) = \sigma \vec{\alpha}(J)$. Thus, we have:

$$\inf \quad \sigma$$

$$\text{s.t} : \sum_{J \subseteq L} \pi_L(J) M_{J,L} (\vec{\beta}(J) - \vec{\gamma}(J)) \leq 0$$

$$\|\vec{\gamma}(J)\| = \sigma \qquad \forall \quad J \subseteq L$$

$$\|\vec{\beta}(J)\| = 1 \qquad \forall \quad J \subseteq L$$

$$\vec{\gamma}(J), \vec{\beta}(J) \geq 0$$

For the above LP, let $(\vec{x}, \{y(J)\}, \{z(J)\})$ denote the dual variables associated with the constraints. The dual is given by

$$\max_{\vec{x}, \{y(J)\}, \{z(J)\}} \min_{\sigma, \vec{\alpha}(J), \vec{\beta}(J)} \sigma +$$

$$\sum_{i=1}^{L} x_i \left( \sum_{J \subseteq L} \pi_L(J) [\sum_{j=1}^{|IS_J|} M_{ij}^J (\beta_j^J - \gamma_j^J)] \right) +$$

$$\sum_{J \subset L} y(J) \left( \vec{\gamma}(J)'e - \sigma \right) +$$

$$\sum_{J \subset L} z(J) \left( \vec{\beta}(J)'e - 1 \right)$$

$$\text{s.t:} \vec{\gamma}(J), \vec{\beta}(J) \geq 0$$

Rewriting the above dual optimization problem, we have

$$\max_{\vec{x}, \{y(J)\}, \{z(J)\}} \min_{\sigma, \vec{\alpha}(J), \vec{\beta}(J)} -\sum_{J} z(J) + \sigma(1 - \sum_{J} y(J)) +$$

$$\sum_{j=1}^{|IS_J|} \beta_j^J \left[ \pi_L(J) \sum_{i=1}^{L} x_i M_{ij}^J + z(J) \right] +$$

$$\sum_{j=1}^{|IS_J|} -\gamma_j^J \left[ \pi_L(J) \sum_{i=1}^{L} x_i M_{ij}^J + y(J) \right]$$

$$\text{s.t:} \vec{\gamma}(J), \vec{\beta}(J) \geq 0$$

Equivalently, the above program can be reduced to

$$\max \sum_{J:J\subseteq L} -z(J)$$

$$\text{s.t} : \pi_L(J)x'M_{J,L} + z(J)e' \geq 0 \quad \forall J \subseteq L$$

$$- \pi_L(J)x'M_{J,L} + y(J)e' \geq 0 \quad \forall J \subseteq L$$

$$\sum_{J\subseteq L} y(J) = 1$$

Denoting $\frac{-z(J)}{\pi(J)}$ by $a(J)$ and $\frac{y(J)}{\pi(J)}$ by $b(J)$ we have the desired result.

∎

From the above Lemma 3, we have that $\sigma_L^*(\boldsymbol{\pi})$ is equal to,

$$\max_{x,a(J),b(J)} \sum_{J:J\subseteq L} \pi_L(J)a(J)$$

$$\text{s.t} : x'M_{J,L} \geq a(J)e' \quad \forall J \subseteq L$$

$$x'M_{J,L} \leq b(J)e' \quad \forall J \subseteq L$$

$$\sum_{J\subseteq L} \pi_L(J)b(J) = 1$$

Observe that $(\frac{1}{\sum \pi_L(J)N(M_J)}e, \frac{n(M_J)}{\sum \pi_L(J)N(M_J)}, 1)$ is a valid point in the search space. Substituting the point in the above function, we have the desired inequality.

∎

*Corollary 1:* $\sigma_G^*(\boldsymbol{\pi}) \geq \frac{1}{d_I(G)}$

*Proof:* Observing the fact that $n(M_J) \geq \frac{1}{d_I(G)}N(M_J)$ and using the above lemma, we have the desired inequality.

∎

**Theorem (4).** *Under a given network topology and channel state distribution with Assumption A1 on the arrivals and fading channels, GMS can stabilize the network if the arrival rates are inside the region $\Lambda_f(\vec{x})$, where $x(S) = \frac{1}{d_I(S)}$.*

*Proof:* We consider a continuous model similar to the one described in the proof of Theorem 1b. In this model, the queuing system evolves according to the following equation,

$$\frac{d}{dt}q_l(t) = \begin{cases} \lambda_l - \mu_l(t) & \text{if} \quad q_l(t) > 0 \\ (\lambda_l - \mu_l(t))^+ & \text{if} \quad q_l(t) = 0, \end{cases}$$

where $\mu_l(t) = \sum_J \pi(J)\mu_l^J(t)$ satisfies the GMS properties. In the original system with fading channels note that the weight of GMS schedule is always greater than $\frac{1}{d_I(S)}$ of the weight of the max-weight schedule where $S$ is the set of links that are in 'ON' state. Therefore in the fluid model, we can show that $\mu_l^J(t)$ satisfies the following condition

$$\sum_l q_l(t)\mu_l^J(t) \geq \frac{1}{d_I(J)} \max_{\vec{\eta}_J \in \mathcal{CH}(M_{J,\kappa})} \sum_l q_l(t)\eta_J(l).$$

Let us the consider the following Lyapunov function,

$$V(\vec{q}(t)) = \sum_l q_l^2(t). \tag{29}$$

Taking the derivate of the Lyapunov function, we have that

$$\dot{V}(\vec{q}(t)) \leq 2 \sum_l q_l(t)(\lambda_l) - \mu_l(t)). \tag{30}$$

Using the GMS properties of $\mu_l(t)$, we have

$$\dot{V}(\vec{q}(t)) \leq \left( 2 \sum_l q_l(t)\lambda_l - \sum_J \frac{2}{d_I(J)} \pi(J) \max_{\vec{\eta}_J \in \mathcal{CH}(M_{J,\mathcal{K}})} \sum_l q_l(t)\eta_J(l) \right) \tag{31}$$

As $\vec{\lambda}$ is assumed to lie inside the region $\Lambda_f(\vec{x})$, there exists $\vec{\eta}_J \in \mathcal{CH}(M_{J,\mathcal{K}})$ such that

$$\lambda_l < \sum_J \frac{1}{d_I(J)} \pi(J)\eta_J(l). \tag{32}$$

Using the above inequality, we have that

$$\dot{V}(\vec{q}(t)) < \left( 2 \sum_l q_l(t) \sum_J \frac{1}{d_I(J)} \pi(J)\eta_J(l) - \sum_J \frac{2}{d_I(J)} \pi(J) \max_{\vec{\eta}_J \in \mathcal{CH}(M_{J,\mathcal{K}})} \sum_l q_l(t)\eta_J(l) \right) \tag{33}$$

Thus from the above inequality we have that $\dot{V}(q(t)) < 0$ whenever $q(t) > 0$.

We can now use the results from [2] to argue that the original system is stable under the assumed arrival process as the fluid model is stable. ∎

## V. EXTENSIONS TO MULTIPLE FADING STATES

We now extend our results for 'ON/OFF' channels to channel models where each link capacity is time-varying and takes values from a finite state space. Let us denote the set of values in the state space by $\{0, c_1, c_2, ....., c_m\}$. The global state $GS(t)$ of the system now refers to the exact channel state of each link. Let $\pi(X_1, X_2, ..., X_K)$ denote the fraction of time the network is in global channel state $(X_1, X_2, X_3, ....X_K)$. Let us denote the state $(X_1, X_2, X_3, ....., X_K)$ by $\mathbf{X}$.

Let $M_{\mathbf{X}}$ denote the matrix consisting of $K$ rows one for each link. Each column now represents a possible maximal independent set on the set of links with non-zero channel states. For a given column, the entries of a given row is set to zero if link $l$ (corresponding to row) does not belong to independent set, or is set to equal to channel value $X_l$ if it belongs to independent set. For example, consider the Interference graph in Figure 1 with each link taking 3 channel states $\{0, 1, 2\}$. Then $M_{(1,2,1,0)}$ is given by,

$$M_{(1,2,1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The throughput region $\Lambda_f$ for the above general network model with fading pattern $\pi(\mathbf{X})$ is given by:

$$\Lambda_f^g = \{\vec{\lambda} : \vec{\lambda} > 0 \quad , \quad \vec{\lambda} \leq \sum_{\mathbf{X}} \pi(\mathbf{X})\vec{\eta}_{\mathbf{X}} \text{ where}$$
$$\vec{\eta}_{\mathbf{X}} \in \mathcal{CH}(M_{\mathbf{X}})\}.$$

We now define the F-LPF for a set of links $L$ as follows:

$$\sigma_L^*(\boldsymbol{\pi}) = \inf\{\sigma : \exists \vec{\phi_1}, \vec{\phi_2} \in \Phi^g(L) \text{ such that } \sigma\vec{\phi_1} \geq \vec{\phi_2}\}, \tag{34}$$

where,

$$\Phi^g(L) = \{\vec{\phi} : \vec{\phi} = \sum_{\mathbf{X}} \pi(\mathbf{X})\vec{\eta}_{\mathbf{X}} \text{ where } \vec{\eta}_{\mathbf{X}} \in \mathcal{CH}(M_{\mathbf{X_L}})\}, \tag{35}$$

$\mathbf{X_L}$ is constructed from $\mathbf{X}$ by setting the values of links that do not belong to set $L$ in $\mathbf{X}$ to zero.

Theorem 1 can be shown to hold for the general model with the above modified definition of F-LPF. The proof of Theorem 1 for the 'ON/OFF' channels can be easily modified to above system with general channels and is therefore omitted.

## VI. CONCLUSION & DISCUSSION

In this paper, we studied the problem of scheduling in wireless networks with interference constraints where the capacity of links changes over time. We have analyzed the performance of a well-known algorithm, Greedy-Maximal Scheduling (GMS), to the case of general wireless networks with fading structure. We defined Fading-Local pooling factor for graphs with fading and showed that it characterizes the fraction of throughput that can be achieved by GMS. We have derived useful yet easily computable bounds on F-LPF through alternate formulations.

By analyzing F-LPF, we have studied the effect of fading on the performance of GMS. It is a priori not clear whether fading can enhance/degrade the relative performance of GMS. In this work, we have showed that fading can in fact exhibit both behaviors through two simple examples, one in which fading increases the efficiency ratio of GMS and other in which fading decreases the efficiency ratio as compared to non-fading case.

## REFERENCES

[1] Berk Birand, Maria Chudnovsky, Bernard Ries, Paul Seymour, Gil Zussman, and Yori Zwols. Analyzing the performance of greedy maximal scheduling via local pooling and graph theory. In *Proc. IEEE Infocom.*, San Diego, California, March 2010.

[2] J. G. Dai. On positive harris recurrence of multiclass queueing networks: A unified approach via fluid limit models. *The Annals of Applied Probability*, 5(1):49–77, 1995.

[3] Antonis Dimakis and Jean Walrand. Sufficient conditions for stability of longest-queue-first scheduling: Second-order properties using fluid limits. *Advances in Applied Probability*, 38(2):505–521, 2006.

[4] L. Georgiadis, M.J. Neely, and L. Tassiulas. Resource allocation and cross-layer control in wireless networks. *Foundations and Trends in Networking*, 1(1):1–144, 2006.

[5] C. Joo, X. Lin, and N. B. Shroff. Understanding the capacity region of the greedy maximal scheduling algorithm in multi-hop wireless networks. *IEEE/ACM Transactions on Networking*, 17(4):1132–1145, August 2009.

[6] Changhee Joo and Ness Shroff. Performance of random access scheduling schemes in multihop wireless networks. In *Proc. IEEE Infocom.*, 2007.

[7] Long Bao Le, Eytan Modiano, Changhee Joo, and Ness B. Shroff. Longest-queue-first scheduling under sinr interference model. In *Proceedings of the eleventh ACM international symposium on Mobile ad hoc networking and computing*, MobiHoc '10, pages 41–50, 2010.

[8] Mathieu Leconte, Jian Ni, and Rayadurgam Srikant. Improved bounds on the throughput efficiency of greedy maximal scheduling in wireless networks. In *Proceedings of the tenth ACM international symposium on Mobile ad hoc networking and computing*, MobiHoc '09, 2009.

[9] Bo Li, Cem Boyaci, and Ye Xia. A refined performance characterization of longest-queue-first policy in wireless networks. In *Proceedings of the tenth ACM international symposium on Mobile ad hoc networking and computing*, MobiHoc '09, pages 65–74, New York, NY, USA, 2009.

[10] Qiao Li and Rohit Negi. Greedy maximal scheduling in wireless networks. In *GLOBECOM'10*, pages 1–5, 2010.

[11] X. Lin and S. Rasool. Constant-time distributed scheduling policies for ad hocwireless networks. In *Proc. Conf. on Decision and Control*, 2006.

[12] Nick McKeown. *Scheduling Algorithms for Input-Queued Cell Switches*. PhD Thesis, University of California at Berkeley,, Berkeley, CA, 1995.

[13] A. Reddy, S. Sanghavi, and S. Shakkottai. On the effect of channel fading on greedy scheduling. In *Proceedings of IEEE Infocom*, Orlando, FL, 2012.

[14] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Trans. Automat. Contr.*, 4:1936–1948, December 1992.

[15] G. Zussman and A. Brzezinski; E. Modiano;. Multihop local pooling for distributed throughput maximization in wireless networks. In *Proc. IEEE Infocom.*, Phoenix, Arizona, April 2008.

## VII. APPENDIX

*Proof:* (Lemma 2)

The proof is similar to the one presented in [5] however taking in to account the channel fading. From the definition of set $L_0(t)$ in Eqn (23), there exists $\epsilon_1 > 0$ such that

$$q_i(t) > q_j(t) + \epsilon_1 \quad \forall\, i \in L_0(t) \text{ and } j \in \mathcal{K} \setminus L_0(t).$$

Using the continuous property of $q_l(t)$, we further have that, there exists $\epsilon_2 > 0, \delta_1 > 0$ such that

$$\min_{i \in L_0(t)} q_i(t + \delta) > \max_{j \in \mathcal{K} \setminus L_0(t)} q_j(t + \delta) + \epsilon_2 \,\forall\, \delta \in [0, \delta_1].$$

Since $L(t)$ is contained inside $L_0(t)$, we have that, there exists $\epsilon_2 > 0, \delta_1 > 0$ such that

$$\min_{i \in L(t)} q_i(t + \delta) > \max_{j \in \mathcal{K} \setminus L_0(t)} q_j(t + \delta) + \epsilon_2 \,\forall\, \delta \in [0, \delta_1]. \tag{36}$$

Also, from the definition of set $L(t)$ in Eqn (24), there exists $\epsilon_3 > 0$ such that

$$\frac{d}{dt} q_i(t) > \frac{d}{dt} q_j(t) + \epsilon_3 \quad \forall\, i \in L(t) \text{ and } j \in L_0(t) \setminus L(t).$$

Further, using the definition of derivative $\frac{d}{dt}q(t) \approx \frac{q(t+\delta)-q(t)}{\delta}$, there exists $\epsilon_4 > 0, \delta_2 > 0$ such that the following holds. For all $i \in L(t)$ and $j \in L_0(t) \setminus L(t)$, we have

$$\frac{q_i(t+\delta) - q_i(t)}{\delta} > \frac{q_j(t+\delta) - q_j(t)}{\delta} + \epsilon_4 \,\forall\, \delta \in (0, \delta_2]$$

Using the fact that queues $q_l(t)$ in set $L_0(t)$ are equal, the above inequality can be rewritten as follows. For all $i \in L(t)$ and $j \in L_0(t) \setminus L(t)$, we have

$$\frac{q_i(t+\delta)}{\delta} > \frac{q_j(t+\delta)}{\delta} + \epsilon_4 \,\forall\, \delta \in (0, \delta_2].$$

Thus we have,

$$\min_{i \in L(t)} q_i(t+\delta) > \max_{j \in L_0(t) \setminus L(t)} q_j(t+\delta) + \epsilon_5 \,\forall\, \delta \in (0, \delta_2]. \tag{37}$$

From the inequalities (36) and (37), we have the following inequality, there exists $\delta_0, \delta_3 > 0$ such that for all $\delta \in [\delta_0, \delta_3]$ we have

$$\min_{i \in L(t)} q_i(n(t+\delta)) > \max_{j \in \mathcal{K} \setminus L(t)} q_j(n(t+\delta)) + \epsilon_6. \tag{38}$$

From the definition of fluid limit $q_l(t)$, there exists $n_0$ large enough such that $\forall n > n_0$ and $\delta \in [\delta_0, \delta_3]$, we have that

$$\min_{i \in L(t)} Q_i(n(t+\delta)) > \max_{j \in \mathcal{K} \setminus L(t)} Q_j(n(t+\delta)) + n\epsilon_7. \tag{39}$$

The above inequality ensures that the links in the set $L(t)$ have larger queue lengths compared to other links in the network for all the time slots in $[n(t+\delta_0), n(t+\delta_3)]$. Therefore, depending up on global channel state $GS(\tau)$, at each time slot $\tau \in [n(t+\delta_0), n(t+\delta_3)]$, GMS schedule picks a maximal schedule from the set of links $L(t)$ that are in 'ON' state. Let $Z_l^n(\tau)$ denote the scheduling decision picked by the GMS algorithm for link $l$ at time slot $\tau$. We thus have

$$\vec{Z}^n(\tau)|_{L(\tau)} \in M_{GS(\tau) \cap L(t), L(t)}. \tag{40}$$

Computing the total service provided by the GMS algorithm in time slots $[n(t+\delta_0), n(t+\delta_3)]$, we have

$$D_l^n(nt + n\delta_3) - D_l^n(nt + n\delta_0) = \int_{nt+n\delta_0}^{nt+n\delta_3} Z_l^n(\tau)d\tau.$$

Let us denote the quantity $\frac{D_l^n(nt+n\delta_3) - D_l^n(nt+n\delta_0)}{n(\delta_3 - \delta_0)}$ by $\mu_l^n(t)$. From the above equality, we have that $\vec{\mu}^n(t)|_{L(t)} \in \Phi(L(t))$. As $\delta_0$ can be made arbitrarily small, we have the result.

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